Appendix S1: Proofs

[*Throughout the appendix we use the shorthand notation* $P_{r|i}$ *for* $P_{return|i}$.]

Proof of Lemma 1

We first show that $h_{MTO}(\delta)$ is quasiconcave when $\alpha \geq v_o/p$, and quasiconvex when $\alpha < v_o/p$. We use the following result from Mangasarian (*Nonlinear Programming*, 1969. McGraw-Hill, New York, p.148): *The function* $h(\cdot) = \frac{g(\cdot)}{\gamma(\cdot)}$ *is quasiconcave (quasiconvex) on a set* $\Gamma \subset \mathbb{R}^n$ *if* $g(\cdot)$ *is concave (convex) on* Γ, $\gamma(\cdot) > 0$ *on* Γ, *and* $\gamma(\cdot)$ *is linear on* $\mathbb{R}^n$.

We show concavity (convexity) of $g(\delta)$ by examining its second derivative: $g''(\delta) = -\lambda(\alpha p - v_o)\left[2P'_{r|\delta} + \delta P''_{r|\delta}\right]$ where $P'_{r|\delta} = -\frac{\mu_1}{\mu_2}P_{r|\delta}\delta^{-1}$ and $P''_{r|\delta} = \frac{\mu_1}{\mu_2}\left(\frac{\mu_1}{\mu_2}+1\right)P_{r|\delta}\delta^{-2}$. The term in brackets simplifies to $2P'_{r|\delta} + \delta P''_{r|\delta} = \frac{\mu_1}{\mu_2}\left(\frac{\mu_1}{\mu_2}-1\right)P_{r|\delta}\delta^{-1} > 0$ (since $\mu_1 \geq \mu_2$). We can therefore state that $g(\delta)$ is concave when $\alpha \geq v_o/p$, and convex when $\alpha < v_o/p$. Since $\gamma(\delta)$ is strictly positive and linear in δ, it follows that $h_{MTO}(\delta)$ is quasiconcave when $\alpha \geq v_o/p$, and quasiconvex when $\alpha < v_o/p$.

Next, we show that $h_{MTO}(\delta)$ is non-decreasing when $\alpha \geq v_o/p$. To do so, we evaluate the first derivative of $h_{MTO}(\delta)$. For convenience, let $\widehat{S} = S \cup \{\delta\}$.

$$h'_{MTO}(\delta) = -\lambda P_0^{\widehat{S}} \sum_{j\in S} M_j P_j^{\widehat{S}} + \lambda P_0^{\widehat{S}} M_\delta \left(\sum_{j\in S} P_j^{\widehat{S}} + P_0^{\widehat{S}}\right) - \lambda(\alpha p - v_o) P_\delta^{\widehat{S}} P'_{r|\delta} \tag{2}$$

It is easy to see that $M_\delta = p - c - (\alpha p - v_o)P_{r|\delta}$ is increasing in δ since $P_{r|\delta}$ is decreasing in δ. Therefore, there must exist some $\widehat{\delta}$ such that $M_{\widehat{\delta}} \geq M_j$, $\forall j \in S$. Then, for all $\delta \geq \widehat{\delta}$, the first term in $h'_{MTO}(\delta)$ is less than the second term, and their sum is therefore positive. The last term is nonnegative since $P'_{r|\delta} < 0$. Hence, we can conclude that in the interval $[\widehat{\delta}, \infty)$, $h'_{MTO}(\delta)$ is positive, or $h_{MTO}(\delta)$ is increasing when $\alpha \geq v_o/p$.

On the other hand, for $\delta < \widehat{\delta}$, we can show by contradiction that the function $h_{MTO}(\delta)$ is non-decreasing. Let δ_L and δ_H be two values of δ such that $\delta_L < \widehat{\delta} < \delta_H$, and assume that $h_{MTO}(\delta)$ is decreasing for $\delta < \widehat{\delta}$. Then, $h_{MTO}(\delta_L) > h_{MTO}(\widehat{\delta})$, and $h_{MTO}(\widehat{\delta}) < h_{MTO}(\delta_H)$.

Furthermore, by the definition of quasiconcavity: $h_{MTO}(\Delta\delta_L + (1-\Delta)\delta_H) \geq \min\{h_{MTO}(\delta_L), h_{MTO}(\delta_H)\}$, with $\Delta \in [0,1]$. Since $\delta_L < \widehat{\delta} < \delta_H$, there exists $\widehat{\Delta} \in [0,1]$ such that $\widehat{\Delta}\delta_L + (1-\widehat{\Delta})\delta_H = \widehat{\delta}$. Using the definition of quasiconcavity for $\Delta = \widehat{\Delta}$, we obtain $h_{MTO}(\widehat{\delta}) \geq \min\{h_{MTO}(\delta_L), h_{MTO}(\delta_H)\}$, which contradicts $h_{MTO}(\delta_L) > h_{MTO}(\widehat{\delta})$, or $h_{MTO}(\widehat{\delta}) < h_{MTO}(\delta_H)$. Hence, when $\alpha \geq v_o/p$, we conclude that $h_{MTO}(\delta)$ is non-decreasing for $\delta < \widehat{\delta}$ as well.

Proof of Lemma 2

Product $i \in N \setminus S$ should be added to the current assortment S *iff* $\Pi_{MTO}(\widehat{S}) \geq \Pi_{MTO}(S)$, where $\widehat{S} = S \cup \{i\}$, or $\lambda \left(P_0^S - P_0^{\widehat{S}} \right) M_i \geq \sum_{j \in S} \lambda \left(P_j^S - P_j^{\widehat{S}} \right) (M_j - M_i) + f$. This inequality states that the profit gain made by the additional market share captured by adding product i should be larger than the potential profit loss due to cannibalization plus the fixed cost. Using the N-MNL purchase probabilities, we rewrite it as follows:

$$\left[\frac{1}{1 + \sum_{k \in S} \omega_k} - \frac{1}{1 + \sum_{k \in \widehat{S}} \omega_k} \right] M_i \geq \sum_{j \in S} \left[\frac{\omega_j}{1 + \sum_{k \in S} \omega_k} - \frac{\omega_j}{1 + \sum_{k \in \widehat{S}} \omega_k} \right] (M_j - M_i) + f/\lambda$$

The result follows from this inequality by further algebraic manipulations.

Proof of Theorem 1

Part (a), the lenient return policy case. The proof is by construction. Suppose the optimal assortment S^*_{MTO} has cardinality k, with $k \in \{1, ..., n-1\}$. (The theorem holds trivially for $|S^*_{MTO}| = 0$ and n.) Take *any* subset S of N with cardinality k. Let $n_a = \max\{i \mid \mathcal{A}_i \subseteq S\ ,\ \ i \in \{0, 1, ..., k\}\}$ be the number of most popular products of N that belong to S. Clearly n_a cannot be strictly larger than k. If $n_a < k$, then there must exist some product $j \in S$ such that $j > n_a + 1$. Since $h_{MTO}(\delta)$ is non-decreasing due to Lemma 1, product j can be replaced with product $n_a + 1$ without decreasing the profit. Proceeding recursively with such profit-improving replacements, $n_a = k$ will in the end be satisfied, which implies that $S^*_{MTO} = \mathcal{A}_k$.

Part (b), the strict return policy case. The proof is by construction. Suppose the optimal assortment S^*_{MTO} has cardinality $k \in \{1, ..., n-2\}$. (The theorem holds trivially for $|S^*_{MTO}| = 0$, $|S^*_{MTO}| = n-1$ and $|S^*_{MTO}| = n$.) Take *any* subset S of N with cardinality k. Let $n_a = \max\{i \mid \mathcal{A}_i \subseteq S\ ,\ \ i \in \{0, 1, ..., k\}\}$ be the number of most popular products of N that belong to S; and $n_z = \max\{j \mid \mathcal{Z}_j \subseteq S\ ,\ \ j \in \{0, 1, ..., k\}\}$ be the number of most eccentric products of N that belong to S. Clearly $n_a + n_z$ cannot be strictly larger than k. If $n_a + n_z < k$, then there must exist some product $i \in S$ such that $n_a + 1 < i < n - n_z$. Since $h_{MTO}(\delta)$ is quasiconvex due to Lemma 1, product i can be replaced with product $n_a + 1$ or with product $n - n_z$ without decreasing the profit. Proceeding recursively with such profit-improving replacements, $n_a + n_z = k$ will in the end be satisfied, which implies that $S^*_{MTO} = \mathcal{A}_j \cup \mathcal{Z}_{k-j}$ for some $j \in \{0, 1, ..., k\}$.

Proof of Lemma 3

Product $i \in N \setminus S$ should be added to the current assortment S *iff* $\Pi_{MTS}(\widehat{S}) \geq \Pi_{MTS}(S)$, where $\widehat{S} = S \cup \{i\}$, or $\sum_{j \in S} \lambda P_j^{\widehat{S}} \widetilde{M}_j^{\widehat{S}} + \lambda P_i^{\widehat{S}} \widetilde{M}_i^{\widehat{S}} - f(|S| + 1) \geq \sum_{j \in S} \lambda P_j^S \widetilde{M}_j^S - f|S|$. Rearranging the terms and dividing by λ, we get $P_i^{\widehat{S}} \widetilde{M}_i^{\widehat{S}} \geq \sum_{j \in S} \left[P_j^S \widetilde{M}_j^S - P_j^{\widehat{S}} \widetilde{M}_j^{\widehat{S}} \right] + f/\lambda$. This is equivalent to $P_i^{\widehat{S}} \widetilde{M}_i^{\widehat{S}} \geq \sum_{j \in S} \widetilde{M}_j^{\widehat{S}} \left[P_j^S - P_j^{\widehat{S}} \right] + \sum_{j \in S} \left[\widetilde{M}_j^S - \widetilde{M}_j^{\widehat{S}} \right] P_j^S + f/\lambda$. Substituting the purchase probabilities into the first term of the right-hand-side, and dividing both sides by $P_i^{\widehat{S}}$, we obtain the desired inequality.

Proof of Lemma 4

We begin by noting that the last two terms of the condition given in Lemma 3 are strictly positive ($\widetilde{M}_j^S > \widetilde{M}_j^{\widehat{S}}$, because $P_j^S > P_j^{\widehat{S}}$ and $(P_j^S)^{\beta-1} < (P_j^{\widehat{S}})^{\beta-1}$ for all $j \in S$). Therefore, if product i improves the expected profit when added to S, it must satisfy $\widetilde{M}_i^{\widehat{S}} > \sum_{j\in S} P_j^S \widetilde{M}_j^{\widehat{S}}$. Now, multiplying the left-hand-side of this inequality by $\left(\sum_{j\in S} P_j^{\widehat{S}} + P_0^{\widehat{S}}\right)$ and the right-hand-side by an equivalent term $\left(\frac{1+\sum_{j\in S}\omega_j}{1+\omega_i+\sum_{j\in S}\omega_j}\right)$, and applying further algebraic manipulations, we obtain the result.

Proof of Lemma 5

We first show that $h_{MTS}(\delta)$ is quasiconvex when $\alpha < v_o/p$ using the same result due to Mangasarian (1969) stated in the first part of the proof of Lemma 1. We already know from that proof that $g(\delta)$ is convex when $\alpha < v_o/p$. We need to show that $\tilde{g}(\delta)$ is also convex when $\alpha < v_o/p$. Taking the first and second derivatives, we verify below that this is indeed the case for any α.

$$\tilde{g}'(\delta) = -(e-v_n)\sigma\lambda^\beta\phi(z^*)\left[\beta\delta^{\beta-1}\gamma(\delta)^{1-\beta} + \left(\sum_{j\in S}\omega_j^\beta + \delta^\beta\right)(1-\beta)\gamma(\delta)^{-\beta}\right]$$

$$\tilde{g}''(\delta) = (e-v_n)\frac{\beta(1-\beta)\sigma\lambda^\beta\phi(z^*)}{\gamma(\delta)^\beta}\left[\frac{\delta^{\beta-2}\left(1+\sum_{j\in S}\omega_j\right)^2 + \sum_{j\in S}\omega_j^\beta}{\gamma(\delta)}\right] > 0$$

Since both $g(\delta)$ and $\tilde{g}(\delta)$ are convex when $\alpha < v_o/p$, their sum is also convex; and, due to Mangasarian's result, $h_{MTS}(\delta)$ is quasiconvex when $\alpha < v_o/p$.

Next, we show that $h_{MTS}(\delta)$ is increasing when $\alpha \geq v_o/p$ as long as δ is profit-improving. To do so, we let $\widehat{S} = S \cup \{\delta\}$, and evaluate the derivative of $h_{MTS}(\delta)$:

$$h'_{MTS}(\delta) = -\lambda(\alpha p - v_o)P_\delta^{\widehat{S}}P'_{r|\delta} - \lambda P_0^{\widehat{S}}\sum_{j\in S}\left[p - c - (\alpha p - v_o)P_{r|j} - (e-v_n)\sigma\phi(z^*)\beta\left(\lambda P_j^{\widehat{S}}\right)^{\beta-1}\right]P_j^{\widehat{S}}$$
$$+\lambda P_0^{\widehat{S}}\left[p - c - (\alpha p - v_o)P_{r|\delta} - (e-v_n)\sigma\phi(z^*)\beta\left(\lambda P_\delta^{\widehat{S}}\right)^{\beta-1}\right]\left(\sum_{j\in S}P_j^{\widehat{S}} + P_0^{\widehat{S}}\right)$$

Note that the terms in square brackets are similar to the expected profit margin per unit sales, except for the value of β multiplying the inventory cost term. Using the identity $\widetilde{M}_i^{\widehat{S}} = M_i - (e-v_n)\sigma\phi(z^*)\left(\lambda P_i^{\widehat{S}}\right)^{\beta-1}$ and applying a few algebraic manipulations:

$$h'_{MTS}(\delta) = \lambda\beta\left[-(\alpha p - v_o)P_\delta^{\widehat{S}}P'_{r|\delta} - P_0^{\widehat{S}}\sum_{j\in S}\widetilde{M}_j^{\widehat{S}}P_j^{\widehat{S}} + P_0^{\widehat{S}}\widetilde{M}_\delta^{\widehat{S}}\left(\sum_{j\in S}P_j^{\widehat{S}} + P_0^{\widehat{S}}\right)\right]$$
$$+\lambda(1-\beta)\left[-(\alpha p - v_o)P_\delta^{\widehat{S}}P'_{r|\delta} - P_0^{\widehat{S}}\sum_{j\in S}M_jP_j^{\widehat{S}} + P_0^{\widehat{S}}M_\delta\left(\sum_{j\in S}P_j^{\widehat{S}} + P_0^{\widehat{S}}\right)\right]$$

Using the derivative of h_{MTO} given in (2) and rearranging the terms, $h'_{MTS}(\delta)$ is:

$$h'_{MTS}(\delta) = \lambda\beta P_0^{\widehat{S}}\left[\widetilde{M}_\delta^{\widehat{S}}\left(\sum_{j\in S}P_j^{\widehat{S}} + P_0^{\widehat{S}}\right) - \sum_{j\in S}\widetilde{M}_j^{\widehat{S}}P_j^{\widehat{S}}\right] - \lambda\beta(\alpha p - v_o)P_\delta^{\widehat{S}}P'_{r|\delta} + (1-\beta)h'_{MTO}(\delta)$$

We know from Lemma 1 that $h'_{MTO}(\delta)$ is non-negative. The second term is also non-negative since $P'_{r|\delta} < 0$ and $\alpha \geq v_o/p$. Assuming that the new product with preference δ is profit-improving, thus using the necessary condition stated in Lemma 4, the remaining term is also positive.

Proof of Theorem 2

Part (a), the lenient return policy case. The proof is by construction and analogous to the proof of Theorem 1a. We only comment on the technical assumption of Lemma 5 that requires the new product to be profit-improving for $h_{MTS}(\delta)$ to be increasing. If no such product exists, then the current assortment must be optimal. Else, it is easy to show that the product with maximum attractiveness (hence, the maximum value of δ) also improves the expected profit (in fact, maximally so). Assume product $i \in N \setminus S$ with preference $\delta = \omega_i$ improves the profit when added. The derivative of $h_{MTS}(\delta)$ evaluated at ω_i is positive (by Lemma 5). Then a product with preference $\delta = \omega_i + \Delta$, for any $\Delta > 0$, would also be profit-improving because $h_{MTS}(\omega_i + \Delta) > h_{MTS}(\omega_i) > \Pi_{MTS}(S)$. We could repeat the same reasoning and keep increasing Δ until $\omega_i + \Delta = \max_{j \in N \setminus S}(\omega_j)$.

Part (b), the strict return policy case. Since the proof is analogous to the proof of Theorem 1b, we only give a sketch of the argument. Due to the quasiconvexity of $h_{MTS}(\delta)$ when $\alpha < v_o/p$, a fact provided by Lemma 5, any assortment that does not conform to the structure stated in the theorem can be improved by pairwise interchanges, and therefore cannot be optimal.

Proof of Proposition 1

Take an MTO environment with a lenient return policy, and suppose the most profitable assortment S has cardinality $k < n$. By Theorem 1, $S = \mathcal{A}_k$. So, for the greedy algorithm to produce the globally optimal assortment, we need to establish that once the most popular of the remaining products fails to satisfy the necessary condition in Lemma 2, no more products can be added and increase the expected profit. Assume that product $k+1$ violates the necessary condition in Lemma 2, and therefore cannot be included in S to improve the profit. By Lemma 1, no product i with $k+1 \leq i \leq n$ improves the profit either, i.e., $M_i < \sum_{j \in S} P_j^S M_j + f / \left(\lambda P_i^{S \cup \{i\}}\right)$. Is it possible then to add two or more products at once and increase the profit? Let $B \subseteq N \setminus S$ be this subset of products to add. If $S \cup B$ improves the profit, then

$$\sum_{j \in S} \lambda P_j^{S \cup B} M_j - f|S| + \sum_{k \in B} \lambda P_k^{S \cup B} M_k - f|B| > \sum_{j \in S} \lambda P_j^S M_j - f|S|$$

$$\sum_{k \in B} \lambda P_k^{S \cup B} M_k - f|B| > \sum_{j \in S} \lambda \left(P_j^S - P_j^{S \cup B}\right) M_j = \left(\sum_{k \in B} P_k^{S \cup B}\right) \sum_{j \in S} \lambda P_j^S M_j$$

Let i be the most popular product in B. The above inequality implies $M_i > \sum_{j\in S} P_j^S M_j + f|B|/\left(\lambda \sum_{k\in B} P_k^{S\cup B}\right)$. Also considering $\sum_{k\in B} P_k^{S\cup B} \leq |B| P_i^{S\cup B}$ and $P_i^{S\cup\{i\}} > P_i^{S\cup B}$, we obtain

$$M_i > \sum_{j\in S} P_j^S M_j + \frac{f|B|}{\lambda \sum_{k\in B} P_k^{S\cup B}} \geq \sum_{j\in S} P_j^S M_j + \frac{f}{\lambda P_i^{S\cup B}} > \sum_{j\in S} P_j^S M_j + \frac{f}{\lambda P_i^{S\cup\{i\}}}$$

which contradicts the initial premise that product i could not be added to S and improve the profit. MTO environment with no returns is a special case with $d = +\infty$.

Proof of Proposition 2

By Theorem 1a, $S^*_{MTO} = \mathcal{A}_k$ for some $k \in \{0, 1, \ldots, n\}$. In addition, by Lemma 2, when $f = 0$ the firm adds product $(i+1)$ to an existing assortment $S = \mathcal{A}_i$ *iff* $\left[1 + \sum_{j\in S} \omega_j\right] M_{i+1} \geq \sum_{j\in S} \omega_j M_j$. Note that the expected profit margin M_j is decreasing and preference ω_j is increasing in α. Therefore, if M_{i+1} decreases at least as fast as M_j for all $j \in S$ as α increases, the left-hand-side will be relatively smaller compared to the right-hand-side. Equivalently: the above inequality will be less likely to be satisfied; a more lenient return policy (higher α) within the lenient region ($\alpha \geq v_o/p$) will lead to lower variety. For this to be true, it is sufficient that $\left|\frac{\partial M_{i+1}}{\partial \alpha}\right| \geq \left|\frac{\partial M_j}{\partial \alpha}\right|$ for all $j \in S$. We have

$$\frac{\partial M_j}{\partial \alpha} = -\frac{\partial P_{r|\,j}}{\partial \alpha}(\alpha p - v_o) - p P_{r|\,j} = \; -p P_{r|\,j}\left[1 + \frac{1}{\mu_2}(\alpha p - v_o)(1 - P_{r|\,j})\right]$$

where the last equality is obtained by substituting $\frac{\partial P_{r|\,j}}{\partial \alpha} = \frac{p}{\mu_2} P_{r|\,j}(1 - P_{r|\,j})$.

Thus, the inequality $\left|\frac{\partial M_{i+1}}{\partial \alpha}\right| \geq \left|\frac{\partial M_j}{\partial \alpha}\right|$ is equivalent to

$$P_{r|\,i+1} + P_{r|\,i+1}\frac{1}{\mu_2}(\alpha p - v_o)(1 - P_{r|\,i+1}) \geq P_{r|\,j} + P_{r|\,j}\frac{1}{\mu_2}(\alpha p - v_o)(1 - P_{r|\,j}).$$

Since $P_{r|\,i+1} \geq P_{r|\,j}$ for all $j \in \mathcal{A}_i$, the condition above is always satisfied if $P_{r|\,i+1} \leq 0.5$, or equivalently $a_{i+1} \geq \alpha p - d$. Hence, the condition that ensures all products to have a positive expected utility, i.e., $a_j \geq p$ for all $j \in N$, is sufficient for the above inequality to hold.

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